CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough.

CIP seminar,
Friday conversations:
For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

CCRT[16] Higher order BTT (part 2).

Asymptotic conditions and character properties.

Disclaimer. – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

1. Magnus groups for various monoids
2. Some subgroups of the Magnus group
3. Paths drawn on the Magnus group and subgroups.
4. Local analysis
5. Integration and Picard's process
6. Analysis of the classical BTT
7. todo: from now Definition of evolution equations
8. Computations with differential modules
9. Some concluding remarks

Introduction

0. Today, we will use the same analysis/synthesis method as in CCRT[16] (part one) and use the information gathered to consider solutions of the BTT as paths drawn on the Magnus group w.r.t. character properties.

1. The mental process for the making of the BTT [10] with various conditions will be the following

$$\underbrace{\text{Differentiation: observing the tangent space} \rightarrow \text{Integration} \rightarrow}_{\textit{Analysis}}$$

$$\underbrace{\text{Technical condition} \rightarrow \text{ NSC} \rightarrow \text{ Proof \& Boundaries}}_{\textit{Synthesis / Integration}}$$

2. This method is not new, it is that of Archimedes (-287, -212) [1], Liu Hui (220-280) [15] and Cavalieri (1598-1647) [6]. Archimedes work was originally thought to be lost, but in 1906 was rediscovered in the celebrated Archimedes Palimpsest.

Iterated integrals and shuffle properties

3. We have seen last time BTT and its link with iterated integrals (initial condition at some $z_0 \in \Omega$).
4. Today, we will see why, integrating it, we have shuffle properties (i.e. the generating series being a shuffle character) and how this property can survive to variations (asymptotic initial conditions, other mutipliers).
5. Let us start with the datum of
    1. a domain $\Omega \subset \mathbb{C}$, a base point $z_0 \in \Omega$, an alphabet X (both non-empty)
    2. a family of "inputs" $(u_x)_{x \in X}$ $u_x \in \mathcal{H}(\Omega)$
6. The system $S' = (\sum_{x \in X} u_x\, x)S \ ;\ S(z_0) = 1_{\mathbb{C}\langle\langle X \rangle\rangle}$ has a unique solution given by the limit of Picard's process

$$S_0 = 1_{X^*} \ ;\ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n$$

Iterated integrals and shuffle properties/2

7. Existence of the limit of Picard's process is due to the fact that the multiplier is without constant term (and then this is a general fact, true with all $M \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$).
8. Now, we need to construct a sort of tensor product adapted to linear forms.
9. In fact, we will need to compute with the monoid of bi-words $X^* \otimes X^*$ because, if $S \in \mathcal{A}\langle\langle X\rangle\rangle$, then $\Delta_{ш}(S) \in \mathcal{A}\langle\langle X^* \otimes X^*\rangle\rangle$, where $\Delta_{ш}(S)$ is the double series

$$\Delta_{ш}(S) = \sum_{w \in X^*} \overbrace{\langle S|w\rangle \underbrace{\sum_{I+J=[|w|]} w[I] \otimes w[J]}_{\Delta_{ш}(w)}}^{Summable\ ?} \qquad (1)$$

Iterated integrals and shuffle properties/3

10. Today, the monoid of bi-words, i.e. $X^* \otimes X^* \simeq X^* \times X^*$, defined by

$$(u_1 \otimes v_1)(u_2 \otimes v_2) = (u_1 u_2 \otimes v_1 v_2) \tag{2}$$

will be sufficient.

11. It is the class of monoids $\mathcal{M} = 1_M + \mathcal{M}_+$ such that[a]

$$\bigcap_{n \geq 1} (\mathcal{M}_+)^n = \emptyset \tag{3}$$

12. Note that $(\mathcal{M}_+)^n$ is the set of $m \in \mathcal{M}$ which can be factorized in n non-trivial factors i.e. $m = u_1 \cdots u_n \; ; \; u_i \neq 1$.

13. Equation (3) gives rise to the length function

$$l(m) = sup\{n \in \mathbb{N} \mid m \in (\mathcal{M}_+)^n\} \text{ with } (\mathcal{M}_+)^0 = \mathcal{M}$$

14. It satisfies $l^{-1}(0) = 1_{\mathcal{M}} \; ; \; l(uv) \geq l(u) + l(v)$.

[a] Locally finite monoids, see [13].

Mini theory of LF-NCDE

15 Let $\mathcal{M}$ be a locally finite (LF) monoid, every NCDE

$$S' = M.S \ ; \ \langle S | 1_M \rangle = 1_\Omega \tag{4}$$

where $S \in \mathcal{H}(\Omega)\langle\langle\mathcal{M}\rangle\rangle$ and $M \in \mathcal{H}(\Omega)_+\langle\langle\mathcal{M}\rangle\rangle$. Admit solutions (indeed even a unique solution) such that

$$S(z_0) = 1_\Omega 1_{X^*} = 1_{\mathcal{H}(\Omega)\langle\langle\mathcal{M}\rangle\rangle} \tag{5}$$

16 Picard's process

$$S_0 = 1_{X^*} \ ; \ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n \tag{6}$$

converges to a series $S_{z_0}^{Pic}$ and the set of all solutions of (4) is the orbit

$$S_{z_0}^{Pic} \ . \ \underbrace{(1 + \mathbb{C}_+\langle\langle\mathcal{M}\rangle\rangle)}_{\text{Galois group of (4).}} \tag{7}$$

Mini theory of LF-NCDE/2

17 Now, we will apply this to the BTT (fuschian type, all $a_x \notin \Omega$)

$$\left\{\begin{array}{rcl} \mathbf{d}(S) & = & M.S \text{ with } M = \sum_{x\in X} \frac{\lambda_x x}{z-a_x} \;;\; \lambda_x \neq 0 \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{array}\right. \quad (8)$$

18 **Remark** $1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} = 1_{\mathcal{H}(\Omega)}.1_{X^*}$

19 At each point $z \in \Omega$, $S(z) \in \mathbb{C}\langle\langle X\rangle\rangle$, so,

1. We can (always) compute $\Delta_{ш}(S(z)) \in \mathbb{C}\langle\langle X^* \otimes X^*\rangle\rangle$
2. Due to the fact that $\mathbb{C}$ is s field, the natural arrow $\mathbb{C}\langle\langle X\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}\langle\langle X\rangle\rangle \hookrightarrow \mathbb{C}\langle\langle X^* \otimes X^*\rangle\rangle$ is into.
3. We have the equivalences
    1. S is a character of $(\mathcal{H}(\Omega)\langle X\rangle, ш, 1_{X^*})$.
    2. For all $z \in \Omega$, $S(z)$ is a character of $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$.
4. We will see next time, that the group of characters of $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ is a closed subgroup of the Magnus group $1_{X^*} + \mathbb{C}_+\langle\langle X\rangle\rangle$.

Mini theory of LF-NCDE/3

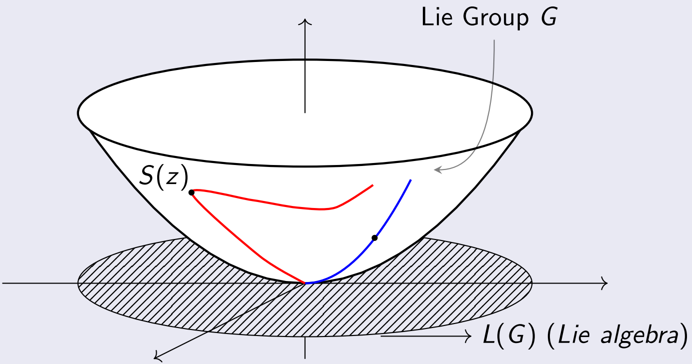
20 **Remark**: We have implicitely used the one-to-one identification

`Functions to series` $\longrightarrow$ `Series of functions`

here

$$\mathcal{A}\langle\langle X\rangle\rangle^{\Omega}\longrightarrow\mathcal{A}^{\Omega}\langle\langle X\rangle\rangle \tag{9}$$

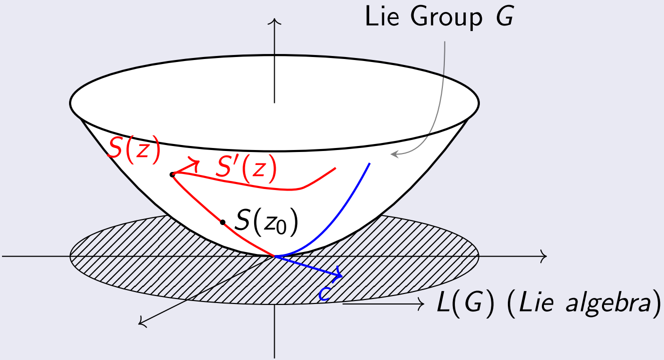
21 From this, we see $S(z)$ as a path drawn on some closed subgroup.



Mini theory of LF-NCDE/4

22. The paradigm we will use in the future is that, if $S(z)$ (each coordinate holomorphic), drawn on the Magnus group is such that
    1. $S(z_0)$ belongs to some closed subgroup G
    2. $\mathbf{d}(S)S^{-1}[z] = M(z)$ belongs, for all $z \in \Omega$ to the tangent space $T_1(G)$.

    Then, $S(z)$ is entirely drawn on G.



Our first subgroup: shuffle characters

23. For today we will content ourselves with the Hausdorff group of the Hopf algebra $(\mathbb{C}\langle X\rangle, ш, 1_{X^*}, \Delta_{\texttt{conc}}, \epsilon)$ (the antipode exists but is not needed here). Let us recall its features
    1. The shuffle product between two words is defined by recursion or duality (see our paper [9])
    2. $\Delta_{\texttt{conc}}$, the dual of conc is defined, within $\mathbb{C}\langle X\rangle$, by duality
    $$\langle \Delta_{\texttt{conc}}(w)|u \otimes v\rangle = \langle w|uv\rangle$$
    or combinatorially $\Delta_{\texttt{conc}}(w) = \sum_{uv=w} u \otimes v$
    3. $\epsilon(P) = \langle P|1_{X^*}\rangle$
24. For every Hopf algebra $(\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \epsilon)$, the set $\Xi(\mathcal{B})$ of characters of $(\mathcal{B}, \mu, 1_{\mathcal{B}})$ is a group under convolution (a monoid in case of a general bialgebra, see our paper [12] Prop. 5.6).
25. Here, due to the fact that $\mathbb{C}$ is a field, we can characterize the group of shuffle characters $\Xi(\mathcal{B})$ by the (algebraic) equations

$$\langle S|1_{X^*}\rangle = 1_{\mathbb{C}} \ ; \ \Delta_{ш}(S) = S \otimes S \tag{10}$$

Our first subgroup: shuffle characters/2

26 Let us now consider an evolution equation $S' = M.S$ in $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ with a primitive multiplier, i.e. for all $z \in \Omega$
$\Delta_{ш}(M(z)) = M(z) \otimes 1_{X^*} + 1_{X^*} \otimes M(z)$

27 Then, if S is group-like (for $\Delta_{ш}$) at one point $z_0 \in \Omega$, it is group-like everywhere (we will see that the point can be remote).

28 Let us have a look at the proof, from which we will deduce the version with asymptotic initial condition. We propose the first following statement

Proposition

Let be given, within $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$, the following evolution equation

$$S' = M.S\ ;\ S(z_0) = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \tag{11}$$

we suppose that, for all $z \in \Omega$, $M(z)$ is primitive (for $\Delta_{ш}$).

Then, *for all $z \in \Omega$, $S(z)$ is group-like (for $\Delta_{ш}$). This, in view of Slide 9, means that S is a character of $(\mathcal{H}(\Omega)\langle X\rangle, ш, 1_{X^*})$.*

Our first subgroup: shuffle characters/3

Proof

29. Firstly, we transform (11) by $\Delta_{ш}$ (which commute - easy exercise - with derivation)

$$\Delta_{ш}(S)' = \Delta_{ш}(S') = \Delta_{ш}(M).\Delta_{ш}(S)\ ;\ \Delta_{ш}(S(z_0)) = 1 \otimes 1$$

30. Taking into account that M is primitive, we get

$$\Delta_{ш}(S)' = (M \otimes 1 + 1 \otimes M).\Delta_{ш}(S)\ ;\ \Delta_{ш}(S(z_0)) = 1 \otimes 1 \quad (12)$$

31. Let us see what happens to $S \otimes S$

$$(S \otimes S)' \overset{(1)}{=} S' \otimes S + S \otimes S' = MS \otimes S + S \otimes MS = (M \otimes 1 + 1 \otimes M).(S \otimes S) \quad (13)$$

32. We see that $\Delta_{ш}(S)$ and $S \otimes S$ satisfy the same evolution equation (same multiplier) and same initial condition (at z_0).

Our first subgroup: shuffle characters/4

Proof

33. Then, for every $z \in \Omega$, we have $\Delta_{ш}(S(z)) = S(z) \otimes S(z)$ (and still $\langle S(z) | 1_{X^*} \rangle = 1_{\mathbb{C}}$).
34. Finally, by the last remark of slide 9, we get that S is a character of $(\mathcal{H}(\Omega)\langle X \rangle, ш, 1_{X^*})$.

Let us try this one.

35. As an excellent exploratory exercise, we can try the multiplier
$$u_0.x_0 + u_1.x_1 + u_2.[x_0, x_1]$$
with $u_i \in \mathcal{H}(\Omega)$.
36. For example, with
$u_0 = 1/z,\ u_1 = 1/(1-z),\ u_2 = \big(2\,\mathrm{Li}_2 + \log(z)\log(1-z)\big)'$ we do not have linear independence of $(\langle S|w \rangle)_{w \in X^*}$.

What is the condition ?

First questions and apps.

37 **Questions.** –
Q1) What are the applications of this situation ?
Q2) What are the consequences ?
Q3) Can we enlarge the scope of this statement to other bialgebras ?

Applications (Q1)

38 All classical (degree-one) multipliers $M = \sum_{x \in X} u_x\, x$ and also

39 Sums of Lie polynomials (with holomorphic inputs/weights) $M = \sum_{x \in X} u_x\, P_x$, where $(P_x)_{x \in X}$ is summable.

Shuffle characters/5

Consequences

40. Within the differential algebra $\mathcal{A} = (\mathcal{H}(\Omega), \frac{d}{dz})$, we have the following

Proposition (A)

Let $\Omega \subset \mathbb{C}$ be a domain, X be an alphabet. We consider the evolution equation on the Magnus group $1_{X^*} + \mathbb{C}_+\langle\langle X\rangle\rangle$, $S' = MS;\ S(z_0) = 1_{\mathbb{C}\langle\langle X\rangle\rangle}$ with multiplier $M = \sum_{x\in X} u_x\, x \in (\mathcal{C}\langle\langle X\rangle\rangle)_1$ ($\mathcal{C}$ is a differential subalgebra of $\mathcal{A}$).
Then, S is a shuffle character and, moreover, TFAE (with $S = \sum_{w\in X^*} \alpha_{z_0}^z(w)\, w$)

1. *The morphism $(\mathcal{C}\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*}) \to (\mathrm{span}_{\mathcal{C}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}, \times, 1_\Omega)$ is injective.*
2. *$\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}$ is $\mathcal{C}$-linearly independent.*
3. *$\{\alpha_{z_0}^z(l)\}_{l\in\mathcal{L}yn\mathcal{X}}$ is $\mathcal{C}$-algebraically independent.*
4. *$\{\alpha_{z_0}^z(x)\}_{x\in\mathcal{X}}$ is $\mathcal{C}$-algebraically independent.*
5. *$\{\alpha_{z_0}^z(x)\}_{x\in\mathcal{X}\cup\{1_{\mathcal{X}^*}\}}$ is $\mathcal{C}$-linearly independent.*

Shuffle characters/6

Widening

41. Setting G, the group of characters of $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$. We can change, in Eq. (11), the condition $S(z_0) = 1$ for $S(z_0) = g$ with $g \in G$ (the trick is to consider $T = S.g^{-1}$).
42. Likewise, we can replace the condition $S(z_0) = 1$ by an asymptotic one like $\lim_{z\to z_0} S(z)R(z)^{-1} = T$ where R, a counter-term, is a character of $(\mathcal{H}(\Omega)\langle X\rangle, ш, 1_{X^*})$ and T a character $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$. Then S is a character of $(\mathcal{H}(\Omega)\langle X\rangle, ш, 1_{X^*})$.
43. This property can be used for evolution equation as

$$S' = (\frac{x_0}{z} + \frac{x_1}{1-z})S \; ; \; \lim_{z\to 0} S(z)e^{-x_0\log(z)} = 1_{\mathcal{H}(\Omega)\langle X\rangle} \tag{14}$$

this equation has a unique solution and the reader must be aware that, contrariwise to those with $S(z_0) = 1$ (with $z_0 \in \Omega$ where there is always a solution, an Evolution equation of type (14) might have no solution.

Shuffle characters/7

Widening

44. Indeed, as BTT holds for solutions of $S' = M.S$; $\langle S|1_M\rangle = 1_\Omega$ (Magnus group condition), we have the following

Proposition (B)

Let $\Omega \subset \mathbb{C}$ be a domain, X be an alphabet, and S be a solution of the evolution equation (on the Magnus group $1_{X^*} + \mathbb{C}_+\langle\langle X\rangle\rangle$), $S' = MS$; $\langle S|1_{X^*}\rangle = 1_{\mathcal{H}(\Omega)}$ with multiplier $M = \sum_{x\in X} u_x\, x \in (\mathcal{C}\langle\langle X\rangle\rangle)_1$ ($\mathcal{C}$ is a differential subalgebra of $\mathcal{A}$). Then, if S is a shuffle character (global or at one point), TFAE

1. *The morphism $(\mathcal{C}\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*}) \to (\mathrm{span}_{\mathcal{C}}\{\langle S|w\rangle\}_{w\in\mathcal{X}^*}, \times, 1_\Omega)$ is injective.*
2. *$\{\langle S|w\rangle\}_{w\in\mathcal{X}^*}$ is $\mathcal{C}$-linearly independent.*
3. *$\{\langle S|l\rangle\}_{l\in\mathcal{L}yn\mathcal{X}}$ is $\mathcal{C}$-algebraically independent.*
4. *$\{\langle S|x\rangle\}_{x\in\mathcal{X}}$ is $\mathcal{C}$-algebraically independent.*
5. *$\{\langle S|y\rangle\}_{y\in\mathcal{X}\cup\{1_{\mathcal{X}^*}\}}$ is $\mathcal{C}$-linearly independent.*

Concluding remarks

45. We have set the problem of evolution equations on the Magnus group with left mutiplier (LM-NCDE).
46. We have seen that **every regular path** drawn on this group is a solution of a (LM-NCDE).
47. In particular, when the multiplier is primitive (for $\Delta_{\text{Ш}}$) and a solution is group-like at one point (or asympotically so), then it is unique and group-like everywhere (a particular case of the closed subgroup property, see a forthcoming CCRT).
48. As a particular case of primitive multipliers, we have the (finite or infinite) sums of letters weighted with "inputs" taken within a differential subfield (and - forthcoming with localization - domain subalgebra). For these multipliers we have the BTT and, for group-like (shuffle character) solutions, we get very strong properties for the algebraic independence of coordinates w.r.t. the differential subalgebra/field.

THANK YOU FOR YOUR ATTENTION !

[1] Archimedes method of Mechanical Theorems https://en.wikipedia.org/wiki/The_Method_of_Mechanical_Theorems

[2] J. Berstel and C. Reutenauer, *Noncommutative Rational Series with Applications*, Cambridge University Press, March 2013 (Online), 2009 (Print).

[3] N. Bourbaki, *Theory of sets*, Springer-Verlag Berlin Heidelberg 2004

[4] N. Bourbaki.– *Algebra ch 1-3*, Springer-Verlag Berlin and Heidelberg GmbH & Co. K; (2nd printing 1989)

[5] N. Bourbaki.– *Commutative Algebra*, Hermann (1972)

[6] Cavalieri's principle https://en.wikipedia.org/wiki/Cavalieri's_principle

[7] Jean Dieudonné, *Infinitesimal calculus*, Houghton Mifflin (1971)

[8] G. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak, *One-Parameter Groups and Combinatorial Physics*, Proceedings of the Symposium COPROMAPH3 : Contemporary Problems in Mathematical Physics, Cotonou, Benin, Scientific World Publishing (2004). arXiv: quant-ph/0401126

[9] G.H.E. Duchamp, J.Y. Enjalbert, Hoang Ngoc Minh, C. Tollu, *The mechanics of shuffle products and their siblings*, Discrete Mathematics 340(9): 2286-2300 (2017).

[10] M. Deneufchâtel, G.H.E. Duchamp, V. Hoang Ngoc Minh and A. I. Solomon, *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

[11] G. H. E Duchamp (LIPN), N. Gargava (EPFL), Hoang Ngoc Minh (LIPN), P. Simonnet (SPE), *A localized version of the basic triangle theorem.*, r arXiv:1908.03327.

[12] G.H.E. Duchamp, D. Grinberg, V. Hoang Ngoc Minh, *Three variations on the linear independence of grouplikes in a coalgebra*, arXiv:2009.10970 [math.QA]

[13] S. Eilenberg, *Automata, languages and machines, vol A*. Acad. Press, New-York, 1974.

[14] Lam, Tsit Yuen, *Exercises in modules and rings*, Problem Books in Mathematics, New York: Springer (2007)

[15] Liu Hui, *The Nine Chapters on the Mathematical Art*, `https://en.wikipedia.org/wiki/The_Nine_Chapters_on_the_Mathematical_Art`, ch 5 *Shanggong - Figuring for construction. Volumes of solids of various shapes.*

[16] N. Matthes, P. Lochak, L. Schneps, *Elliptic multizetas and the elliptic double shuffle relations*, Int. Math. Res. Not. IMRN 2021, no. 1, 695–753

[17] M. Lothaire.– *Combinatorics on words*, Cambridge University Press (1997)

[18] Nils Matthes, *On the algebraic structure of iterated integrals of quasimodular forms*, Algebra & Number Theory Vol. 11 (2017), No. 9, 2113-2130 (arXiv:1708.04561)

[19] Christophe Reutenauer, *Free Lie Algebras*, Université du Québec a Montréal, Clarendon Press, Oxford (1993)

[20] Van der Put, Marius, Singer, Michael F., *Galois Theory of Linear Differential Equations*, Springer; (2002)

[21] M. van der Put, *Recent work on differential Galois theory*, Séminaire N. Bourbaki, 1997-1998, exp. n o 849, p. 341-367.

[22] Ore condition
https://en.wikipedia.org/wiki/Ore_condition

[23] Ore localization
https://ncatlab.org/nlab/show/Ore+domain

[24] Constants in localizations.
https://math.stackexchange.com/questions/2051634

[25] Find a power series representation for $2/(2+x)$.
https://math.stackexchange.com/questions/4060489

[26] Hilbert's fifth problem.
https://en.wikipedia.org/wiki/Hilbert's_fifth_problem

[27] About the arrow "can" (tensor product of series $\to$ double series, see MO questions 200442 and 201753), its usage in C.S. and combinatorics II.